Three–Way Tables
Edpsy/Psych/Soc 589

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Outline

■ Types of association
  1. Marginal & Partial tables.
     (a) Marginal Independence and Conditional Dependence.
     (b) Marginal Dependence and Conditional Independence.
     (c) Marginal and Conditional Dependence.
  4. Homogeneous association.

■ Inference for Large Samples.
  2. Estimating common odds ratio.
  4. Comments.

■ Inference for Small Samples (a few comments).
Examples of 3–Way Tables

- Smoking × Breathing × Age.
- Group × Response × Z (hypothetical).
- Boys Scouts × Delinquent × SES (hypothetical).
- Cal graduate admissions × gender × Department.
- Supervisor Job satisfaction × Worker Job satisfaction × Management quality.
- Race × Questions regarding media × Year.
- Employment status × Residence × Months after hurricane Katrina.
3–Way Contingency Table

Slices of this table are “Partial Tables”.

There are 3–ways to slice this table up.
1. $K$ Frontal planes or $XY$ for each level of $Z$.
2. $J$ Vertical planes or $XZ$ for each level of $Y$.
3. $I$ Horizontal planes or $YZ$ for each level of $X$. 
Partial Tables & Marginal Tables

e.g., $XY$ tables for each level of $Z$ ...

The Frontal planes of the box are $XY$ tables for each level of $Z$ are **Partial tables**:

\[
\begin{array}{ccc}
Z = 1 & Z = 2 & \ldots Z = K \\
\begin{array}{c}
1 \ldots j \ldots J \\
\end{array} & \\
& \begin{array}{c}
1 \ldots j \ldots J \\
\end{array} & \\
& \begin{array}{c}
1 \ldots j \ldots J \\
\end{array}
\end{array}
\]

The marginal table is obtained by summing across the $K$ levels of $Z$:

\[
\begin{array}{c}
1 \ldots j \ldots J \\
\end{array}
\]

\[
\begin{array}{c}
X_i \\
\end{array}
\]

\[
\begin{array}{c}
Y_j \\
\end{array}
\]

\[
\begin{array}{c}
I \\
\end{array}
\]

\[
\begin{array}{c}
i \\
\end{array}
\]

\[
\begin{array}{c}
\sum_{k=1}^{K} n_{ijk} \\
\end{array}
\]

\[
\begin{array}{c}
\sum_{k=1}^{K} n_{ij} \\
\end{array}
\]

where

\[
n_{ij+} = \sum_{k=1}^{K} n_{ijk}
\]
Conditional or “Partial” Odds Ratios

Notation:

\[ n_{ijk} = \text{observed frequency of the } (i, j, k)\text{th cell.} \]
\[ \mu_{ijk} = \text{expected frequency of the } (i, j, k)\text{th cell.} \]
\[ = n\pi_{ijk} \]

**Conditional Odds Ratios** are odds ratios between two variables for fixed levels of the third variable.

For fixed level of \( Z \), the conditional \( XY \) association given \( k \)th level of \( Z \) is

\[ \theta_{XY}(k) = \frac{\mu_{11k}\mu_{22k}}{\mu_{12k}\mu_{21k}} \]

& more generally

\[ \theta_{ii',jj'}(k) = \frac{n_{ijk}n_{i'j'k}}{n_{i'jk}n_{ij'k}} \]

Conditional odds ratios are computed using the partial tables, and are sometimes referred to as measures of “partial association”.

If \( \theta_{XY}(k) \neq 1 \), then variables \( X \) and \( Y \) are “Conditionally associated”.
Marginal Odds Ratios

are the odds ratios between two variables in the marginal table.

For example, for the $XY$ margin:

$$
\mu_{ij+} = \sum_{k=1}^{K} \mu_{ijk}
$$

and the “Marginal Odds Ratio” is

$$
\theta_{XY} = \frac{\mu_{11} + \mu_{22} +}{\mu_{12} + \mu_{21} +} \quad \text{& more generally } \quad \theta_{ii',jj'} = \frac{\mu_{ij} + \mu_{i'j'} +}{\mu_{i'j} + \mu_{ij'} +}
$$

With sample data, use $n_{ijk}$ and $\hat{\theta}$.

Marginal association can be very different from conditional association.

The marginal odds ratios need not equal the partial (conditional) odds ratios.
Example of Marginal vs Partial Odds Ratios

These data are from a study reported by Forthofer & Lehnen (1981) (Agresti, 1990). Measures on Caucasians who work in certain industrial plants in Houston were recorded.

- Response/outcome variable: breathing test result (normal, not normal).
- Explanatory variable: smoking status (never, current).
- Conditioning variable: age

Marginal Table (ignoring age):

<table>
<thead>
<tr>
<th>Smoking Status</th>
<th>Test Result</th>
<th>Normal</th>
<th>Not Normal</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never</td>
<td>741</td>
<td>38</td>
<td>779</td>
<td></td>
</tr>
<tr>
<td>Current</td>
<td>927</td>
<td>131</td>
<td>1058</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1668</td>
<td>169</td>
<td>1837</td>
<td></td>
</tr>
</tbody>
</table>

Marginal odds ratio: $\hat{\theta} = 2.756$

$H_0 : \theta = 1$ vs $H_A : \theta \neq 1$  $G^2 = 32.382$, $df = 1$, & $p$–value $< .001$. 

- Conditional or “Partial” Odds Ratios
- Marginal Odds Ratios
- Example of Marginal vs Partial Odds Ratios
- Example: Partial Tables
- Marginal and Conditional Associations
- Four Situations
- Marginal Independence/Conditional Dependence
- Marginal Dependence/Conditional Independence
- Conditional Independence
- Example of Conditional Independence: C.A.L
- C.A.L Admissions Data by Department
- 3rd Example of Conditional Independence
- 3rd Example: Partial Tables
- Simpson's Paradox
- (Hypothetical) Example of Simpson's Paradox
- Picture of Simpson's Paradox
- Homogeneous Association
- Homogeneous Association (continued)
### Example: Partial Tables

<table>
<thead>
<tr>
<th>Age &lt; 40</th>
<th>Smoking Status</th>
<th>Test Result</th>
<th>Odds Ratio</th>
<th>$G^2$</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normal</td>
<td>Not Normal</td>
<td>$\hat{\theta}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Never</td>
<td>577</td>
<td>34</td>
<td>1.418</td>
<td>2.489</td>
<td>.115</td>
</tr>
<tr>
<td>Current</td>
<td>682</td>
<td>57</td>
<td>12.38</td>
<td>45.125</td>
<td>&lt; .001</td>
</tr>
<tr>
<td></td>
<td>1259</td>
<td>91</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Compare these odds ratios with the marginal odds ratio:

$$\hat{\theta} = 2.756$$
Marginal and Conditional Associations

- Independence = “No Association”.
- Dependence = “Association”.
- Marginal Independence means that $\theta_{X,Y} = 1$
- Marginal Dependence means that $\theta_{X,Y} \neq 1$
- Conditional Independence means that $\theta_{X,Y}(k) = 1$ for all $k = 1, \ldots, K$.
- Conditional Dependence means that $\theta_{X,Y}(k) \neq 1$ for at least one $k = 1, \ldots, K$.
- Marginal independence does not imply conditional independence.
- Conditional independence does not imply marginal independence.
### Four Situations

<table>
<thead>
<tr>
<th>Situation</th>
<th>Marginal</th>
<th>Conditional</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Independence</td>
<td>Independence</td>
<td>Not interesting</td>
</tr>
<tr>
<td>2</td>
<td>Independence</td>
<td>Dependence</td>
<td>“Conditional Dependence”</td>
</tr>
<tr>
<td>3</td>
<td>Dependence</td>
<td>Independence</td>
<td>“Conditional Independence”</td>
</tr>
<tr>
<td>4</td>
<td>Dependence</td>
<td>Dependence</td>
<td>“Conditional Dependence”</td>
</tr>
</tbody>
</table>

Conditional dependence includes a number of different cases, which we have terms to refer to them:

- Simpson’s paradox.
- Homogeneous association.
- 3–way association.

We’ll take a look at examples of situations 2, 3 and 4 and each of these cases of conditional dependence.
Marginal and Partial Tables

Conditional and Marginal Odds Ratios
- Conditional or “Partial” Odds Ratios
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- Marginal Dependence/Conditional Independence
- Conditional Independence
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- Homogeneous Association (continued)

Marginal Table

<table>
<thead>
<tr>
<th>Group</th>
<th>Response</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

\[ \theta = 1 \]

\[ \log(\theta) = 0 \]

Partial Tables:

<table>
<thead>
<tr>
<th>Group</th>
<th>Response</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

\[ Z = 1 \]

\[ \theta = 1/9 \]

\[ \log(\theta) = -2.197 \]

<table>
<thead>
<tr>
<th>Group</th>
<th>Response</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

\[ Z = 2 \]

\[ \theta = 1 \]

\[ \log(\theta) = 0 \]

<table>
<thead>
<tr>
<th>Group</th>
<th>Response</th>
<th>yes</th>
<th>no</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

\[ Z = 3 \]

\[ \theta = 9 \]

\[ \log(\theta) = 2.197 \]

Association is in opposite directions in tables \( Z = 1 \) and \( Z = 3 \).
This situation and concept is not unique to categorical data analysis.

Conditional independence is very important and is the basis for many models and techniques including

- Latent variable models (e.g., factor analysis, latent class analysis, item response theory, etc.).
- Multivariate Graphical models, which provide ways to decompose models and problems into sub-problems.

Back to categorical data. . . .
### Conditional Independence

#### Hypothetical Example from Agresti, 1990:

**Marginal Table:**

<table>
<thead>
<tr>
<th>Boy Scout</th>
<th>Delinquent</th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>36</td>
<td>364</td>
<td>400</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>60</td>
<td>340</td>
<td>400</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>96</strong></td>
<td><strong>704</strong></td>
<td><strong>800</strong></td>
<td></td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = 0.56 \]

\[ G^2 = 6.882 \]

\[ p\text{-value} = 0.01 \]

**Partial Tables — condition on socioeconomic status**

**SES = Low**

<table>
<thead>
<tr>
<th>Boy Scout</th>
<th>Delinquent</th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>10</td>
<td>40</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>40</td>
<td>160</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>50</strong></td>
<td><strong>200</strong></td>
<td><strong>250</strong></td>
<td></td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = 1.00 \]

**SES = Medium**

<table>
<thead>
<tr>
<th>Boy Scout</th>
<th>Delinquent</th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>18</td>
<td>132</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>18</td>
<td>132</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>36</strong></td>
<td><strong>264</strong></td>
<td><strong>300</strong></td>
<td></td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = 1.00 \]

**SES = High**

<table>
<thead>
<tr>
<th>Boy Scout</th>
<th>Delinquent</th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>8</td>
<td>192</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>No</td>
<td>2</td>
<td>48</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>10</strong></td>
<td><strong>240</strong></td>
<td><strong>250</strong></td>
<td></td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = 1.00 \]
Example of Conditional Independence: \textit{CAL}


- Question: Is there sex discrimination in admission to graduate school?

- The data for two departments (B & C) of the 6 largest are

<table>
<thead>
<tr>
<th>Gender</th>
<th>Admitted</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Female</td>
<td>219</td>
<td>399</td>
</tr>
<tr>
<td>Male</td>
<td>473</td>
<td>412</td>
</tr>
<tr>
<td></td>
<td>692</td>
<td>811</td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = .48 \]
\[ 1/\hat{\theta} = 2.09 \]
\[ 95\% \text{ CI: (.39, .59)} \]

odds(female admitted) = \( 219/399 = .55 \)
odds(male admitted) = \( 473/412 = 1.15 \)
### CAL Admissions Data by Department

#### Department B:

<table>
<thead>
<tr>
<th>Gender</th>
<th>Admitted</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>17</td>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>Male</td>
<td>353</td>
<td>207</td>
<td>560</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>585</td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = 1.25 \]

95% CI: (.53, 2.94)

#### Department C:

<table>
<thead>
<tr>
<th>Gender</th>
<th>Admitted</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>202</td>
<td>391</td>
<td>593</td>
</tr>
<tr>
<td>Male</td>
<td>120</td>
<td>205</td>
<td>325</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>918</td>
</tr>
</tbody>
</table>

\[ \hat{\theta} = .88 \]

95% CI: (.67, 1.17)
3rd Example of Conditional Independence

...Maybe conditional independence... Job satisfaction (Andersen, 1985). These data are from a large scale investigation of blue collar workers in Denmark (1968).

Three variables:
- Worker job satisfaction (Low, High).
- Supervisor job satisfaction (Low, High).
- Quality of Management (Bad, Good).

The Worker $\times$ Supervisor Job Satisfaction (Marginal Table):

<table>
<thead>
<tr>
<th>Supervisor satisfaction</th>
<th>Worker satisfaction</th>
<th>$\hat{\theta} = 1.86$, 95% CI (1.37, 2.52)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Low</td>
<td>162</td>
<td>196</td>
</tr>
<tr>
<td>High</td>
<td>110</td>
<td>247</td>
</tr>
<tr>
<td></td>
<td>272</td>
<td>443</td>
</tr>
</tbody>
</table>

Statistics $\quad df \quad$ Value $\quad$ Prob

$X^2 \quad 1 \quad 17.00 \quad < .001$

$G^2 \quad 1 \quad 17.19 \quad < .001$
### 3rd Example: Partial Tables

Job satisfaction conditional on management quality

<table>
<thead>
<tr>
<th>Supervisor’s satisfaction</th>
<th>Bad Management</th>
<th>Good Management</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worker’s satisfaction</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Low</td>
<td>103</td>
<td>87</td>
</tr>
<tr>
<td>High</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>135</td>
<td>129</td>
</tr>
</tbody>
</table>

\[
\hat{\theta}_{bad} = 1.55 \quad \text{and} \quad 95\% \text{ CI for } \theta_{bad} \text{ is (.90, 1.67)}
\]

\[
\hat{\theta}_{good} = 1.42 \quad \text{and} \quad 95\% \text{ CI for } \theta_{good} \text{ is (.94, 2.14)}
\]

| Statistic | df | Bad Management | | Good Management | |
|-----------|----|----------------|-----------------|-----------------|
|           |    | Value | p-value | Value | p-value |
| \(X^2\)  | 1  | 2.56  | .11  | 2.85  | .09   |
| \(G^2\)  | 1  | 2.57  | .11  | 2.82  | .09   |

We’ll come back to this example....
Simpson's Paradox

The marginal association is in the opposite direction as the conditional (or partial) association.

Consider 3 dichotomous variables: $X$, $Y$, and $Z$ where

- $P(Y = 1|X = 1) = \text{conditional probability } Y = 1 \text{ given } X = 1$,
- $P(Y = 1|X = 1, Z = 1) = \text{conditional probability } Y = 1 \text{ given } X = 1 \text{ and } Z = 1$.

**Simpson's Paradox:**

Marginal: $P(Y = 1|X = 1) < P(Y = 1|X = 2)$

Conditionals: $P(Y = 1|X = 1, Z = 1) > P(Y = 1|X = 2, Z = 1)$

$P(Y = 1|X = 1, Z = 2) > P(Y = 1|X = 2, Z = 2)$

In terms of odds ratios, it is possible to observed the following pattern of marginal and partial associations:

Marginal odds: $\theta_{XY} < 1$; however, Partial odds: $\theta_{XY(1)} > 1$ and $\theta_{XY(2)} > 1$
### (Hypothetical) Example of Simpson’s Paradox

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 1$</td>
<td>50</td>
<td>900</td>
</tr>
<tr>
<td>$X = 2$</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>51</td>
<td>1000</td>
</tr>
</tbody>
</table>

$\theta_{XY}(z=1) = 5.56$ \hspace{1cm} \text{and} \hspace{1cm} \theta_{XY}(z=2) = 19.0$

$\pi_1(x=1,z=1) = 50/950 = .05$ \hspace{1cm} \text{and} \hspace{1cm} \pi_1(x=1,z=2) = 500/505 = .99$

$\pi_2(x=2,z=1) = 1/101 = .01$ \hspace{1cm} \text{and} \hspace{1cm} \pi_2(x=2,z=2) = 500/595 = .84$

#### The $XY$ margin:

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 1$</td>
<td>550</td>
<td>905</td>
</tr>
<tr>
<td>$X = 2$</td>
<td>501</td>
<td>195</td>
</tr>
<tr>
<td></td>
<td>1051</td>
<td>1100</td>
</tr>
</tbody>
</table>

$\theta_{XY} = .237$

$\pi_1 = 550/1455 = .38$

$\pi_2 = 501/696 = .72$
Picture of Simpson’s Paradox

Example of Simpson’s Paradox
Area = number of observations with X= i & Z=k

Marginal Effect

Level of X

Percent Y=1

Z=2

Z=2

Z=1

Z=1

0.5 1.0 1.5 2.0 2.5

Three-Way Tables (continued)
Homogeneous Association

Definition: The association between variables $X$, $Y$, and $Z$ is “homogeneous” if the following three conditions hold:

$$\theta_{XY}(1) = \ldots = \theta_{XY}(k) = \ldots = \theta_{XY}(K)$$
$$\theta_{XZ}(1) = \ldots = \theta_{XZ}(j) = \ldots = \theta_{XZ}(J)$$
$$\theta_{YZ}(1) = \ldots = \theta_{YZ}(i) = \ldots = \theta_{YZ}(I)$$

- There is “no interaction between any 2 variables in their effects on the third variable”.
- There is “no 3–way interaction” among the variables.
- If one of the above holds, then the other two will also hold.
- Conditional independence is a special case of this.

For example,

$$\theta_{YZ}(1) = \ldots = \theta_{YZ}(i) = \ldots = \theta_{YZ}(I) = 1$$
Homogeneous Association (continued)

- There are even simpler independence conditions are that special cases of homogeneous association, but this is a topic for another day.

- When these three conditions (equations) do **not** hold, then the conditional odds ratios for any pair of variables are not equal. Conditional odds ratios differ/depend on the level of the third variable.

- Example of 3–way Interaction — the Age $\times$ Smoking $\times$ Breath test results example.
### Example of Homogeneous Association

**Attitude Toward Media (Fienberg, 1980). “Are radio and TV networks doing a good, fair, or poor job?”**

<table>
<thead>
<tr>
<th>Year</th>
<th>Race</th>
<th>Good</th>
<th>Fair</th>
<th>Poor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1959</td>
<td>Black</td>
<td>81</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>White</td>
<td>325</td>
<td>243</td>
<td>54</td>
</tr>
<tr>
<td>1971</td>
<td>Black</td>
<td>224</td>
<td>144</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>White</td>
<td>600</td>
<td>636</td>
<td>158</td>
</tr>
</tbody>
</table>

\[
\hat{\theta}_{RQ1(1959)} = \frac{(81)(243)}{(325)(23)} = 2.63
\]
\[
\hat{\theta}_{RQ1(1971)} = \frac{(224)(636)}{(600)(144)} = 1.66
\]
\[
\hat{\theta}_{RQ2(1959)} = \frac{(23)(54)}{(243)(4)} = 1.28
\]
\[
\hat{\theta}_{RQ2(1971)} = \frac{(144)(158)}{(636)(24)} = 1.49
\]
\[
\hat{\theta}_{YR(\text{good})} = \frac{(81)(600)}{(325)(224)} = .68
\]
\[
\hat{\theta}_{YR(\text{fair})} = \frac{(23)(636)}{(243)(144)} = .42
\]
\[
\hat{\theta}_{YR(\text{poor})} = \frac{(4)(158)}{(54)(636)} = .48
\]
\[
\hat{\theta}_{YQ1(\text{black})} = \frac{(81)(144)}{(23)(224)} = 2.26
\]
\[
\hat{\theta}_{YQ1(\text{white})} = \frac{(325)(636)}{(600)(243)} = 1.44
\]
\[
\hat{\theta}_{YQ2(\text{black})} = \frac{(23)(24)}{(4)(144)} = .96
\]
\[
\hat{\theta}_{YQ2(\text{white})} = \frac{(243)(158)}{(54)(646)} = 1.10
\]
Statistical Inference & 3–Way Tables

(Large samples)

We’ll focus methods for $2 \times 2 \times K$ tables.

- Test of conditional independence.
- Estimating common odds ratio.
- Test of homogeneous association.
- Further Comments
Sampling Models for 3–Way Tables

Generalizations of the ones for 2–way tables, but there are now more possibilities.

Possible Sampling Models for 3–Way tables:

- **Independent Poisson** variates — nothing fixed, each cell is Poisson.
- **Multinomial counts** with only the overall total \( n \) is fixed.
- Multinomial counts with fixed sample size for each partial table.

For example, the partial tables of \( X \times Y \) for each level of \( Z \), only the total of each of the \( K \) tables is fixed, \( n_{++k} \).

- **Independent binomial (or multinomial) samples** within each partial table.

For example, if \( n_{1+k} \) and \( n_{2+k} \) are fixed in each \( 2 \times 2 \) partial table of \( X \) crossed with \( Y \) for \( k = 1, \ldots, K \) levels of \( Z \), then we have independent binomial samples within each partial table.
Tests of Conditional Independence

Two methods:

■ Sum of test statistics for independence in each of the partial tables to get an overall chi–squared statistic for “conditional independence” — this is the equivalent to a model based test discussed later in course.

■ Cochran-Mantel-Haenszel Test — we’ll talk about this one first.
Cochran-Mantel-Haenszel Test

Example: Cal graduate admission data
- \( X \): Gender (female, male).
- \( Y \): Admission to graduate school (admitted, denied).
- \( Z \): Department to which person applied (6 largest ones, A–F).

So we have a \( 2 \times 2 \times 6 \) table of Gender by Admission by Department.

For each Gender by Admission partial table (condition on department), if we take the row totals \( (n_{1+k} \) and \( n_{2+k} \)) and the column totals \( (n_{+1k} \) and \( n_{+2k} \)) as fixed, then once we know the value of a single cell within the table, we can fill in the rest of the table. For department \( A \):

<table>
<thead>
<tr>
<th>Gender</th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Female</td>
<td>89</td>
<td>(19)</td>
<td>108</td>
</tr>
<tr>
<td>Male</td>
<td>(512)</td>
<td>(313)</td>
<td>825</td>
</tr>
<tr>
<td></td>
<td>601</td>
<td>332</td>
<td>933</td>
</tr>
</tbody>
</table>
Idea Behind the CMH Test

- From discussion of Fisher’s exact test, we know that the distribution of $2 \times 2$ tables with fixed margins is hypergeometric.

- Regardless of sampling scheme, if we consider row and column totals of partial tables as fixed, we can use hypergeometric distribution to compute probabilities.

- The test for conditional association uses one cell from each partial table.

- Historical Note: In developing this test, Mantel and Haenszel were concerned with analyzing retrospective studies of diseases ($Y$). They wanted to compare two groups ($X$) and adjust for a control variable ($Z$). Even though only 1 margin of the data (disease margin, $Y$) is fixed, they analyzed data by conditioning on both the outcome ($Y$) and group margins ($X$) for each level of the control variable ($Z$).
### Statistical Hypotheses

If the null hypothesis of conditional independence is true, i.e.,

$$H_0 : \theta_{XY(1)} = \ldots = \theta_{XY(K)} = 1$$

Then the mean of the \((1,1)\) cell of \(k\)th partial table is

$$\mu_{11k} = E(n_{11k}) = \hat{\mu}_{11k} = n_{++k} \hat{\pi}_{1+k} \hat{\pi}_{+1} = \frac{n_{1+k} n_{+1k}}{n_{++k}}$$

and the variance of the \((1,1)\) cell of the \(k\)th partial table is

$$\hat{\text{Var}}(n_{11k}) = \frac{n_{1+k} n_{2+k} n_{+1k} n_{+2k}}{n_{++k}^2 (n_{++k} - 1)}$$

If the null is false, then we expect that for tables where

- \(\theta_{XY(k)} > 1 \implies (n_{11k} - \mu_{11k}) > 0\)
- \(\theta_{XY(k)} < 1 \implies (n_{11k} - \mu_{11k}) < 0\)
- \(\theta_{XY(k)} = 1 \implies (n_{11k} - \mu_{11k}) \approx 0\)
CMH Test Statistic

Mantel & Haenszel (1959) proposed the following statistic

\[ M^2 = \frac{\left( \sum_k |n_{11k} - \mu_{11k}| - \frac{1}{2} \right)^2}{\sum_k \text{Var}(n_{11k})} \]

If \( H_0 \) is true, then \( M^2 \) is approximately chi-squared with \( df = 1 \).

Cochran (1954) proposed a similar statistic, except that
- He did not include the continuity correction, “\(-1/2\)”.  
- He used a different \( \text{Var}(n_{11k}) \).

The statistic the we will use is a combination of these two proposed statistics, the “Cochran-Mantel-Haenszel” statistic

\[ CMH = \frac{\left[ \sum_k (n_{11k} - \hat{\mu}_{11k}) \right]^2}{\sum_k \text{Var}(n_{11k})} \]

where
- \( \hat{\mu}_{11k} = \frac{n_{1+k}n_{+1k}}{n_{++k}} \)
- \( \hat{\text{Var}}(n_{11k}) = \frac{n_{1+k}n_{2+k}n_{1+k}n_{2+k}}{n_{++k}^2(n_{++k} - 1)} \)
### Properties of the CMH Test Statistic

\[
CMH = \frac{\left(\sum_k (n_{11k} - \mu_{11k})\right)^2}{\sum_k \text{Var}(n_{11k})}
\]

- For large samples, when \(H_0\) is true, CMH has a chi-squared distribution with \(df = 1\).
- If all \(\theta_{XY(k)} = 1\), then CMH is small (close to 0).
  
  Example: SES × Boy Scout × Delinquent. Since \(\hat{\theta} = 1\) for each partial table, if we compute \(CMH\), it would equal 0 and \(p\)-value=1.00.
- If some/all \(\theta_{XY(k)} > 1\), then CMH is large.
  
  Example: Age × Smoking × Breath Test.
  
  Example: \(CAL\) graduate admissions data, Departments (6 versus 5) × Gender × Admission.
- If some/all \(\theta_{XY(k)} < 1\), then CMH is large.
More Properties of the CMH Test Statistic

\[ CMH = \frac{\left( \sum_k (n_{11k} - \mu_{11k}) \right)^2}{\sum_k \text{Var}(n_{11k})} \]

- If some \( \theta_{XY(k)} > 1 \) and some \( \theta_{XY(k)} < 1 \), \( CMH \) test is not appropriate.

Example: Three tables of Group \( \times \) Response (hypothetical “DIF” case).

- The test works well and is more powerful when \( \theta_{XY(k)} \)'s are in the same direction and of comparable size.

Example: Management quality \( \times \) Worker satisfaction \( \times \) Supervisor’s satisfaction.
### Age × Smoking × Breath test results

Example: These data are from a study reported by Forthofer & Lehnen (1981) (Agresti, 1990). Subjects were whites who work in certain industrial plants in Houston.

Partial Tables:

<table>
<thead>
<tr>
<th>Smoking Status</th>
<th>Test Result</th>
<th>Age &lt; 40</th>
<th>Age 40–59</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Normal</td>
<td>Not Normal</td>
<td>Test Result</td>
</tr>
<tr>
<td>Never</td>
<td>577</td>
<td>34</td>
<td>611</td>
</tr>
<tr>
<td>Current</td>
<td>682</td>
<td>57</td>
<td>739</td>
</tr>
<tr>
<td></td>
<td>1259</td>
<td>91</td>
<td>1350</td>
</tr>
</tbody>
</table>

Statistical Hypotheses:

\[ H_0 : \theta_{SB(<40)} = \theta_{SB(40–50)} = 1 \]
\[ H_A : \text{Smoking and test results are conditionaly dependent.} \]
CMH Statistic for Age × Smoking × Breath

<table>
<thead>
<tr>
<th>Age &lt; 40</th>
<th>Age 40–59</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>( \hat{\theta}_1 = 1.418 )</strong></td>
<td><strong>( \hat{\theta}_2 = 12.38 )</strong></td>
</tr>
<tr>
<td>( \hat{\mu}_{111} = \frac{(611)(1259)}{1350} = 569.81 )</td>
<td>( \hat{\mu}_{112} = \frac{(168)(409)}{487} = 141.09 )</td>
</tr>
<tr>
<td>( n_{111} - \hat{\mu}_{111} = 577 - 569.81 = 7.19 )</td>
<td>( n_{112} - \hat{\mu}_{112} = 164 - 141.09 = 22.91 )</td>
</tr>
<tr>
<td>( \text{var}(n_{111}) = \frac{(611)(739)(1259)(91)}{1350^2(1350-1)} = 21.04 )</td>
<td>( \text{var}(n_{111}) = \frac{(168)(319)(409)(78)}{487^2(487-1)} = 14.83 )</td>
</tr>
</tbody>
</table>

\[
CMH = \frac{(7.19 + 22.91)^2}{21.04 + 14.83} = \frac{24.24}{21.04 + 14.83} = 24.24
\]

with \( df = 1 \) has \( p \)-value < .001.
CMH Example: CAL graduate admissions

The null hypothesis of no sex discrimination is

\[ \theta_{GA(1)} = \theta_{GA(2)} = \theta_{GA(3)} = \theta_{GA(4)} = \theta_{GA(5)} = \theta_{GA(6)} = 1 \]

Department A

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>89</td>
<td>19</td>
<td>108</td>
</tr>
<tr>
<td>male</td>
<td>512</td>
<td>313</td>
<td>825</td>
</tr>
<tr>
<td></td>
<td>601</td>
<td>332</td>
<td>933</td>
</tr>
</tbody>
</table>

Department B

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>17</td>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>male</td>
<td>353</td>
<td>207</td>
<td>560</td>
</tr>
<tr>
<td></td>
<td>370</td>
<td>215</td>
<td>585</td>
</tr>
</tbody>
</table>

Department C

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>202</td>
<td>391</td>
<td>593</td>
</tr>
<tr>
<td>male</td>
<td>120</td>
<td>205</td>
<td>325</td>
</tr>
<tr>
<td></td>
<td>322</td>
<td>596</td>
<td>918</td>
</tr>
</tbody>
</table>

Department D

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>131</td>
<td>244</td>
<td>375</td>
</tr>
<tr>
<td>male</td>
<td>138</td>
<td>279</td>
<td>417</td>
</tr>
<tr>
<td></td>
<td>269</td>
<td>523</td>
<td>792</td>
</tr>
</tbody>
</table>

Department E

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>94</td>
<td>299</td>
<td>393</td>
</tr>
<tr>
<td>male</td>
<td>53</td>
<td>138</td>
<td>191</td>
</tr>
<tr>
<td></td>
<td>147</td>
<td>437</td>
<td>584</td>
</tr>
</tbody>
</table>

Department F

<table>
<thead>
<tr>
<th>Gender</th>
<th>admit</th>
<th>deny</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>female</td>
<td>24</td>
<td>317</td>
<td>341</td>
</tr>
<tr>
<td>male</td>
<td>22</td>
<td>351</td>
<td>373</td>
</tr>
<tr>
<td></td>
<td>46</td>
<td>668</td>
<td>714</td>
</tr>
</tbody>
</table>

\[
CMH = \frac{(19.42 + 1.19 - 6.00 + 3.63 - 4.92 + 2.03)^2}{21.25 + 5.57 + 47.86 + 44.34 + 24.25 + 10.75} \\
= (15.36)^2 / 154.02 \\
= 1.53 \\
(p-value = .217)
\]

Department A: \( \hat{\theta}_A = 2.86, G^2 = 17.248, df = 1, p-value < .001. \)

Without Department A: \( CMH = .125, p-value = .724. \)
Example: Table $\times$ Group $\times$ Response

(Hypothetical DIF data)

\[
\begin{array}{ccc}
\text{Group} & \text{yes} & \text{no} \\
\hline
\text{A} & 5 & 15 \\
\text{B} & 15 & 5 \\
\hline
& 20 & 20 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Group} & \text{yes} & \text{no} \\
\hline
\text{A} & 10 & 10 \\
\text{B} & 10 & 10 \\
\hline
& 20 & 20 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Group} & \text{yes} & \text{no} \\
\hline
\text{A} & 15 & 5 \\
\text{B} & 5 & 15 \\
\hline
& 20 & 20 \\
\end{array}
\]

\[
\begin{array}{c}
\theta = 0.11 \\
\hline
\theta = 1.00 \\
\hline
\theta = 9.00 \\
\end{array}
\]

\[
CMH = \frac{((5 - 10) + (10 - 10) + (15 - 10))^2}{\sum_{k=1}^{3} \text{Var}(n_{11k})} \\
= \frac{(-5 + 0 + 5)^2}{\sum_{k=1}^{3} \text{Var}(n_{11k})} \\
= 0
\]

Why is this test a bad thing to do here?

(continued)
### Management $\times$ Supervisor $\times$ Worker

#### Bad Management

<table>
<thead>
<tr>
<th>Supervisor Satisfaction</th>
<th>Worker Job</th>
<th>Worker Job</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>103</td>
<td>87</td>
</tr>
<tr>
<td>High</td>
<td>32</td>
<td>42</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>135</strong></td>
<td><strong>129</strong></td>
</tr>
</tbody>
</table>

#### Good Management

<table>
<thead>
<tr>
<th>Worker Job</th>
<th>Worker Job</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>59</td>
</tr>
<tr>
<td>High</td>
<td>78</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>137</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\hat{\theta}_{bad}$</th>
<th>$\hat{\theta}_{good}$</th>
<th>95% CI for $\theta_{bad}$</th>
<th>95% CI for $\theta_{good}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.55</td>
<td>1.42</td>
<td>(.90, 1.67)</td>
<td>(.94, 2.14)</td>
</tr>
</tbody>
</table>

#### Statistic

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$df$</th>
<th>Value</th>
<th>$p$–value</th>
<th>Statistic</th>
<th>$df$</th>
<th>Value</th>
<th>$p$–value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^2$</td>
<td>1</td>
<td>2.56</td>
<td>.11</td>
<td>$X^2$</td>
<td>1</td>
<td>2.85</td>
<td>.09</td>
</tr>
<tr>
<td>$G^2$</td>
<td>1</td>
<td>2.57</td>
<td>.11</td>
<td>$G^2$</td>
<td>1</td>
<td>2.82</td>
<td>.09</td>
</tr>
</tbody>
</table>

Note: $G^2 = 2.57 + 2.82 = 5.39$ with $df = 2$ has $p$–value = .068.

(continued)
Combining the results from these two tables to test conditional independence yields $G^2 = 2.57 + 2.82 = 5.39$ with $df = 2$ has $p$–value$= .068$.

Conclusion:
$H_O :$ Conditional independence, $\theta_{SW(bad)} = \theta_{SW(good)} = 1$, is a tenable hypothesis.

Since $\hat{\theta}_{bad} \approx \hat{\theta}_{good}$, CMH should be more powerful.

$$CMH = 5.43 \quad \quad p$–value = .021$$

Next steps:
- Estimate the common odds ratio.
- Test for homogeneous association.
Estimating Common Odds Ratio

For a $2 \times 2$ table where $\theta_{XY(1)} = \ldots = \theta_{XY(K)}$, the “Mantel-Haenszel Estimator” of a common value of the odds ratio is

$$\hat{\theta}_{MH} = \frac{\sum_k (n_{11k}n_{22k}/n_{++k})}{\sum_k (n_{12k}n_{21k}/n_{++k})}$$

For the blue-collar worker example, this value is

$$\hat{\theta}_{MH} = \frac{(103)(42)/264 + (59)(205)/448}{(32)(87)/264 + (78)(109)/448}$$

$$= \frac{16.39 + 27.12}{10.55 + 18.98}$$

$$= \frac{43.51}{29.52} = 1.47$$

Which is in between the two estimates from the two partial tables:

$$\hat{\theta}_{bad} = 1.55 \quad \text{and} \quad \hat{\theta}_{good} = 1.42$$
For our example,

\[ 95\% \text{ confidence interval for } \theta \rightarrow (1.06, 2.04) \]

The standard error for \( \hat{\theta}_{MH} \) is complex, so we will rely on SAS/FREQ to get this. When you supply the “CMH” option to the TABLES command, you will get both CMH test statistic and \( \hat{\theta}_{MH} \) along with a 95% confidence interval for \( \theta \).

**SAS output:**

<table>
<thead>
<tr>
<th>Type of Study</th>
<th>Method</th>
<th>Value</th>
<th>95% Confidence Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case-Control (Odds Ratio)</td>
<td>Mantel-Haenszel</td>
<td>1.4697</td>
<td>1.0600</td>
</tr>
<tr>
<td>(Odds Ratio)</td>
<td>Logit</td>
<td>1.4692</td>
<td>1.0594</td>
</tr>
</tbody>
</table>
SAS input & Common Odds Ratio Estimate

DATA sat;
   INPUT manager $ super $ worker $ count;
   LABEL manager='Quality of management'
   super =’Supervisors Satisfaction’
   worker='Blue Collar Workers Satisfaction';

DATALINES;
Bad   Low    Low  103
Bad   Low    High  87
...  ...
Good  High   Low  78
Good  High   High  205

PROC FREQ DATA=sat ORDER=data;
   WEIGHT count;
   TABLES manage*super*worker /nopercent norow nocol chisq cmh;
   run;
Notes Regarding CMH

- If we have homogeneous association, i.e.,

\[ \theta_{XY}(1) = \ldots = \theta_{XY}(K) \]

then \( \hat{\theta}_{MH} \) is useful as an estimate of the this common odds ratio.

- If the odds ratios are not the same but they are at least in the same direction, then \( \hat{\theta}_{MH} \) can be useful as a summary statistic of the \( K \) conditional (partial) associations.

- If there’s a 3-way interaction, it is misleading to use an estimate of the common odds ratio. E.g., Age \( \times \) Smoking \( \times \) Breath test results, we get as a common estimate of the odds ratio

\[ \hat{\theta}_{SB} = 2.57 \]

But the ones from the separate tables are

\[ \hat{\theta}_{SB(<40)} = 1.42 \quad \text{and} \quad \hat{\theta}_{SB(40-59)} = 12.38 \]
Testing Homogeneity of Odds Ratios

- For $2 \times 2 \times K$ tables.
- Since $\theta_{XY(1)} = \ldots = \theta_{XY(K)}$ implies both

$$\theta_{YZ(1)} = \ldots = \theta_{YZ(I)} \quad \text{and} \quad \theta_{XZ(1)} = \ldots = \theta_{XZ(J)}$$

To test for homogeneous association we only need to test one of these, e.g.

$$H_O : \theta_{XY(1)} = \ldots = \theta_{XY(K)}$$

- Given estimated expected frequencies assuming that $H_O$ is true, the test statistic we use is the “Breslow-Day” statistic, which is like Pearson’s $X^2$:

$$X^2 = \sum_i \sum_j \sum_k \frac{(n_{ijk} - \hat{\mu}_{ijk})^2}{\hat{\mu}_{ijk}}$$

- If $H_O$ is true, then the Breslow-Day statistic has an approximate chi-squared distribution with $df = K - 1$. 

Breslow-Day statistic

- We need \( \hat{\mu}_{ijk} \) for each table assuming that the null hypothesis of homogeneous association is true.
- \( \{\hat{\mu}_{11k}, \hat{\mu}_{12k}, \hat{\mu}_{21k}, \hat{\mu}_{22k}\} \), are found such that
- The margins of the table of estimated expected frequencies equal the observed margins; that is,

\[
\begin{array}{c|c}
\hat{\mu}_{11k} & \hat{\mu}_{12k} \\
\hline
\hat{\mu}_{21k} & \hat{\mu}_{22k} \\
\hline
n_{+1k} & n_{+2k}
\end{array}
\]

\[
(\hat{\mu}_{11k} + \hat{\mu}_{12k}) = n_{1+k}
\]

\[
(\hat{\mu}_{21k} + \hat{\mu}_{22k}) = n_{2+k}
\]

- If the null hypothesis of homogeneous association is true, then \( \hat{\theta}_{MH} \) is a good estimate of the common odds ratio. When computing estimated expected frequencies, we want them such that the odds ratio computed on each of the \( K \) partial tables equals the Mantel-Haenszel estimate of the common odds ratio.

\[
\hat{\theta}_{MH} = \frac{\hat{\mu}_{11k}\hat{\mu}_{22k}}{\hat{\mu}_{12k}\hat{\mu}_{21k}}
\]
Breslow-Day statistic

- Computation of the estimated expected frequencies is a bit complex, so we will rely on SAS/FREQ to give us the Breslow-Day statistic. If you have a $2 \times 2 \times K$ table and request “CMH” options with the TABLES command, you will automatically get the Breslow-Day statistic.

- SAS output for manager $\times$ supervisor $\times$ worker is

  Breslow-Day Test for Homogeneity of the Odds Ratios

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Chi-Square</td>
<td>0.0649</td>
</tr>
<tr>
<td>DF</td>
<td>1</td>
</tr>
<tr>
<td>Pr &gt; ChiSq</td>
<td>0.7989</td>
</tr>
</tbody>
</table>

- For this test, your sample size should be relatively large, i.e.,

$$\hat{\mu}_{ijk} \geq 5 \quad \text{for at least 80\% of cells}$$
Examples: Testing Homogeneity of Association

Worker × Supervisor × Management

- $CMH = 5.34$ with $p$–value $= .02$ $\implies$ conditionally dependent.
- The Mantel-Haenszel estimate of common odds ratio
  \[ \hat{\theta}_{MH} = 1.47 \]
  while the separate ones were
  \[ \hat{\theta}_{bad} = 1.55 \quad \text{and} \quad \hat{\theta}_{good} = 1.42 \]

- Now let’s test the homogeneity of the odds ratios
  \[ H_O : \theta_{WS(bad)} = \theta_{WS(good)}. \]
  Breslow-Day statistic $= .065$, $df = 1$, and $p$–value $= .80$. 
Cal Graduate Admissions data

Six of the largest departments:

- $CMH = 1.53$, $df = 1$, $p$–value $= .217 \implies$ gender and admission are conditionally independent (given department).

- Mantel-Haenszel estimate of the common odds ratio
  \[ \hat{\theta}_{GA} = .91 \]
  and the 95% Confidence interval is
  \[ (.772, 1.061). \]

- Now let’s test homogeneity of odds ratios
  \[ H_0 : \theta_{GA(a)} = \theta_{GA(b)} = \theta_{GA(c)} = \theta_{GA(d)} = \theta_{GA(e)} = \theta_{GA(f)} \]
  Breslow-Day statistic $= 18.826$, $df = 5$, $p$–value $= .002$.

What’s going on?
Cal Graduate Admissions data

Drop Department A, which is the only department for which the odds ratio appears to differ from 1.

- $CMH = .125, \text{df} = 1, \ p-value = .724 \implies \text{gender and admission are conditionally independent (given department)}$.

- The Mantel-Hanelszel estimate of the common odds ratio

$$\hat{\theta} = 1.031$$

and the 95% confidence interval for $\theta_{GA}$ is

$$(.870, 1.211)$$

- The test of homogeneity of odds ratios

$$H_0 : \theta_{GA(b)} = \theta_{GA(c)} = \theta_{GA(d)} = \theta_{GA(e)} = \theta_{GA(f)}$$

Breslow-Day statistic $= 2.558, \ \text{df} = 4, \ p-value = .63$.

Conclusion?.
(Hypothetical DIF Example)

- \( CMH = 0.00, \ df = 1, \) and \( p-value = 1.00 \) \( \Rightarrow \) Group and response are independent given \( Z \).

- Mantel-Haenszel estimate of the common odds ratio

\[ \hat{\theta}_{GR} = 1.00 \]

- Test for homogeneity of the odds ratios yields Breslow-Day statistic = 20.00, \( df = 2, \) and \( p-value < .001. \)

- Conclusion?
Year $\times$ Race $\times$ Response to Question

Response to question “Are radio and TV networks doing a good, fair, or poor job?”

<table>
<thead>
<tr>
<th>Year</th>
<th>Race</th>
<th>Good</th>
<th>Fair</th>
<th>Poor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1959</td>
<td>Black</td>
<td>81</td>
<td>23</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>White</td>
<td>325</td>
<td>243</td>
<td>54</td>
</tr>
<tr>
<td>1971</td>
<td>Black</td>
<td>224</td>
<td>144</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>White</td>
<td>600</td>
<td>636</td>
<td>158</td>
</tr>
</tbody>
</table>

We could test for conditional independence, but which variable should be condition on?

- Year and look at Race $\times$ Response to the Question?
- Race and look at Year $\times$ Response to the Question?
- Response to the Question and look at Year $\times$ Race?
Year × Race × Response to Question

- Since the Breslow-Day statistic only works for $2 \times 2 \times K$ tables, to test for homogeneous association we will set up the test for

$$H_0 : \theta_{YR}(\text{good}) = \theta_{YR}(\text{fair}) = \theta_{YR}(\text{poor})$$

even though we are more interested in the odds ratios between Year & Response and Race & Response.

- Breslow-Day statistic = 3.464, $df = 2$, $p$-value = .18.

Note: There is a generalization of CMH for $I \times J \times K$ tables and we can get an estimate of the common odds ratio between Year and Race (i.e., $\hat{\theta}_{MH} = .57$), what we’d really like are estimates of common odds ratios between Year and Question and between Race and Question.
One Last Example: Hurricane Katrina

Reference: http://www.bls.gov/katrina/cpscesquestions.htm

The effects of hurricane Katrina on BLS employment and unemployment data collection.

- Employment status (employed, unemployed, not in labor force).
- Residence (same or different than in August).
- Month data from (October, November)

The data (in thousands):

<table>
<thead>
<tr>
<th></th>
<th>October</th>
<th></th>
<th>November</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Same</td>
<td>Different</td>
<td>Same</td>
<td>Different</td>
</tr>
<tr>
<td>Employed</td>
<td>153</td>
<td>179</td>
<td>204</td>
<td>185</td>
</tr>
<tr>
<td>Unemployed</td>
<td>18</td>
<td>90</td>
<td>29</td>
<td>71</td>
</tr>
<tr>
<td>Not in labor</td>
<td>134</td>
<td>217</td>
<td>209</td>
<td>188</td>
</tr>
</tbody>
</table>

The data (in thousands):
CMH & Breslow-Day

- There is a generalization of CMH for $I \times J \times K$ tables (which SAS/FREQ will perform).
- There is not such a generalization for the Breslow-Day statistic.
- Given that we can get a non-significant result using CMH when there is association in partial tables, you should check to see whether there is homogeneous association or a 3–way association.
- Breslow-Day statistic does not work well for small samples, while the Cochern-Mantel-Haenszel does pretty well.
- A modeling approach handles $I \times J \times K$ tables and can test the same hypotheses.