Random Intercept Models
Edps/Psych/Stat 587

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Outline

- A very simple case of a random intercept model.
- When to use this model
- The model
  - Empty model
  - One explanatory variable
  - Multiple explanatory variables
- SAS and PROC MIXED

Reading: Snijders & Bosker, pp 38–56
A Simple Model

Situation:

\[ j = 1, \ldots, N \] groups
(e.g., schools).

\[ i = 1, \ldots, n_j \] individuals within the groups
(e.g., students).

\[ Y_{ij} \] a numerical response variable
(e.g., math scores).

\[ x_{ij} \] a possible explanatory variable
(e.g., SES).
Simple Level 1 Model

\[ Y_{ij} = \beta_{0j} + \beta_1 x_{1ij} + R_{ij} \]

where

- $\beta_1$ is fixed in the population.
- $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.
- The intercept, $\beta_{0j}$, depends on the group.
Simple Level 2 Model

The intercept, $\beta_{0j}$, can be broken down into two parts:

- An overall or average value of the intercept:
  $$\gamma_{00}$$

- A group dependent part of the intercept:
  $$U_{0j}$$

- The Level 2 Model is
  $$\beta_{0j} = \gamma_{00} + U_{0j}$$
The Linear Mixed Model

Replace $\beta_{0j}$ in the level 1 model by the level 2 model for $\beta_{0j}$:

$$Y_{ij} = \gamma_0 + U_{0j} + \beta_{1i}x_{1ij} + R_{ij}$$

$$= \gamma_0 + U_{0j} + \gamma_{10}x_{1ij} + R_{ij}$$

where $\beta_1 = \gamma_{10}$

(to be consistent with later models).
NELS88 Data

What the model might look like applied to the NELS88 data ($N = 10$ schools):

\[
\text{NELS88, } N = 10, \text{ ANCOVA}
\]
Two cases of the model

\[ Y_{ij} = \gamma_{00} + \gamma_{10} x_{1ij} + U_{0j} + R_{ij} \]

**Case 1:** ANCOVA — If the \( U_{0j} \)'s are fixed parameters in the population (i.e., The \( U_{0j} \)'s are not random but are the group effects).

**Case 2:** Random intercept model (i.e., HLM) — If the \( U_{0j} \)'s are random; that is, groups are a random sample in the population.
## Contrasting ANCOVA & HLM

<table>
<thead>
<tr>
<th><strong>ANCOVA/ANOVA:</strong></th>
<th><strong>Random Coefficients Model:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups are unique and interest is only in the groups observed.</td>
<td>Groups are a sample from a (real or hypothetical) population and</td>
</tr>
<tr>
<td>A Group’s observations provide information about $U_{0j}$, so with small $n_j$, don’t get precise estimates of group effects (large standard errors).</td>
<td>Group effects come from the same distribution, data from all groups is used $\implies$ can do well with small $n_j$.</td>
</tr>
<tr>
<td>In ANCOVA, $U_{0j}$’s represent all the differences between groups (no unexplained variability left).</td>
<td>Synonyms: “random effects” &amp; “unexplained variability” i.e., $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$, groups are “exchangeable”.</td>
</tr>
<tr>
<td>If $U_{0j}$ and $R_{ij}$ are non-normal and/or dependent, random coefficient model may be unreliable.</td>
<td>Can use a different distribution.</td>
</tr>
</tbody>
</table>
Random Intercept Model: no x’s

The baseline/empty/null HLM (no explanatory variables).

A regression model where the intercept is a random variable.

Models for Clustered Data

Level 1:

\[ Y_{ij} = \beta_0 + R_{ij} \]

where

- \( R_{ij} \sim \mathcal{N}(0, \sigma^2) \), and independent.
- The intercept, \( \beta_0 \), is a random variable for which we specify a linear regression model (level 2, e.g., groups).

Level 2:

\[ \beta_0 = \gamma_0 + U_{0j} \]

where

- \( \gamma_0 \) is the intercept (fixed).
- \( U_{0j} \sim \mathcal{N}(0, \tau^2_0) \) and independent.
- \( U_{0j} \) and \( R_{ij} \) are independent.
Levels 1 and 2

Estimation of level 1 and level 2 models should be simultaneous.

Substituting our equation for $\beta_{0j}$ (level 2 model) into the equation for level 1, we get a composite or **linear mixed model**

$$Y_{ij} = \beta_{0j} + R_{ij}$$

$$= \gamma_{00} + U_{0j} + R_{ij}$$

fixed random

and the **marginal model** is

$$Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2))$$
Estimation

\[ Y_{ij} = \beta_{oj} + R_{ij} \]
\[ = \gamma_{00} + U_{0j} + R_{ij} \]
  \underline{\text{fixed}} \quad \underline{\text{random}}

and

\[ Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2)) \]

- This (marginal model) is what is used to fit the model to data and in SAS/MIXED, we specify the linear mixed model.

- To use HLM7 software, you specify the component models (i.e., the level 1 model, and the level 2 models) and the program figures out what the marginal model is and fits it to the data.
Example: Null Model

High School and Beyond: A nationally representative sub-sample from the 1982 High School and Beyond Survey. It includes

- Information on $n_+ = 7185$ students.
- Students are nested within $N = 160$ schools: 90 public and 70 Catholic.
- Samples sizes averaged about $n_j \sim 45$ per school.
Example: Level 1 Level 2 variables

Level 1 Variables: Student level variables

- **Outcome variable:** Math achievement.
- **Explanatory:**
  - Student socioeconomic status (SES) — composite of parental education, occupation and income. This have been standardized such that average equals 0.
  - Student ethnicity (1=minority, 0=not).
  - Gender (1=female, 0=male).

Level 2 Variables: School level variables

- Sector — whether the school is Catholic (= 1) or public (= 0).
- Mean SES — average SES of students w/in a school.
- School enrollment.
- Proportion of students in academic track.
- Disciplinary climate.
- Whether the percentage minority is \( \leq 40\% \) or \( > 40\% \).
Example: the null HLM

\[ Y_{ij} = \text{math achievement.} \]

Groups are schools: \( j = 1, \ldots, 160. \)

Diagram representing the situation:

\[ \text{math} \]
The Hierarchical Null Model

Level 1:

\[ Y_{ij} = \beta_{0j} + R_{ij} \]

(math)_{ij} = \beta_{0j} + R_{ij}

where \( R_{ij} \sim \mathcal{N}(0, \sigma^2) \) and independent.

Level 2:

\[ \beta_{0j} = \gamma_{00} + U_{0j} \]

where \( U_{0j} \sim \mathcal{N}(0, \tau^2_0) \) and independent.

...and \( R_{ij} \) and \( U_{0j} \) are independent.
The Marginal Model:

The **Linear Mixed model**:

\[
  (\text{math})_{ij} = Y_{ij} = \gamma_{00} + U_{0j} + R_{ij}
\]

where \( U_{0j} \) and \( R_{ij} \) are independent and

The **Marginal Model**:

\[
  Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2)).
\]

Implications for data.
Results 1

Some descriptive statistics:

The MEANS Procedure

<table>
<thead>
<tr>
<th>Variable</th>
<th>Label</th>
<th>Mean</th>
<th>StdDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>mathach</td>
<td>Mathematics achievement</td>
<td>12.748</td>
<td>6.878</td>
</tr>
<tr>
<td>ses</td>
<td>standardized student ses</td>
<td>0.000</td>
<td>0.779</td>
</tr>
</tbody>
</table>
The Mixed Procedure

Model Information

Data Set: SASDATA.HSBALL
Dependent Variable: mathach
Covariance Structure: Variance Components
Subject Effect: id
Estimation Method: ML
Residual Variance Method: Profile
Fixed Effects SE Method: Model-Based
Degrees of Freedom Method: Containment
## Results 3

### Dimensions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariance Parameters</td>
<td>2</td>
</tr>
<tr>
<td>Columns in X</td>
<td>1</td>
</tr>
<tr>
<td>Columns in Z Per Subject</td>
<td>1</td>
</tr>
<tr>
<td>Subjects</td>
<td>160</td>
</tr>
<tr>
<td>Max Obs Per Subject</td>
<td>67</td>
</tr>
<tr>
<td>Observations Used</td>
<td>7185</td>
</tr>
<tr>
<td>Observations Not Used</td>
<td>0</td>
</tr>
<tr>
<td>Total Observations</td>
<td>7185</td>
</tr>
</tbody>
</table>
Results 4

Iteration History

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Evaluations</th>
<th>-2 Log Like</th>
<th>Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>48099.73204627</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>47115.82988208</td>
<td>0.00000114</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>47115.81024259</td>
<td>0.00000000</td>
</tr>
</tbody>
</table>

Convergence criteria met.

Covariance Parameter Estimates

| Cov Parm | Subject | Estimate | Error  | Value     | Pr Z > |t| |
|----------|---------|----------|--------|-----------|--------|---|
| Intercept| id      | 8.5490   | 1.0676 | 8.01      | < .0001|
| Residual |         | 39.1488  | 0.6607 | 59.26     | < .0001|
Results 5

Fit Statistics

-2 Log Likelihood 47115.8
AIC (smaller is better) 47121.8
AICC (smaller is better) 47121.8
BIC (smaller is better) 47131.0

Solution for Fixed Effects

<table>
<thead>
<tr>
<th>Effect</th>
<th>Estimate</th>
<th>Error</th>
<th>DF</th>
<th>t Value</th>
<th>Pr &gt;</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>12.6371</td>
<td>0.2436</td>
<td>159</td>
<td>51.88</td>
<td>&lt; .0001</td>
<td></td>
</tr>
</tbody>
</table>
## Summary of Results

### Null Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>SE</th>
<th>Parameter</th>
<th>Value</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{00}$</td>
<td>12.64</td>
<td>.24</td>
<td>$\tau_{0}^{2}$</td>
<td>8.55</td>
<td>1.07</td>
</tr>
<tr>
<td>$\sigma^{2}$</td>
<td>39.15</td>
<td>.66</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Estimate of the intra-class correlation,

$$\hat{\rho}_I = \frac{\hat{\tau}^2}{\hat{\tau}^2 + \hat{\sigma}^2} = \frac{8.55}{8.55 + 39.15} = \frac{8.55}{47.70} = .18$$

and $(\text{math})_{ij} = 12.64$, the overall (total sample) mean,

The overall variance of math scores is $s^2 = (8.55 + 39.15) = 47.70$.

Note $s = 6.873$ and $s^2 = 47.301$—see descriptive statistics, p 18.
Add SES as Level 1 Explanatory?
Some More Schools
Overlay Regressions for all Schools

Math x Homework
Separate Regression Line for Each School
Random Intercept: One $x$

SES to help explain some of the variability of $Y_{ij}$.

The Hierarchical Model:

Level 1:

$$Y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + R_{ij}$$

where $R_{ij} \sim N(0, \sigma^2)$ and independent.

Level 2:

$$\begin{align*}
\beta_{0j} &= \gamma_{00} + U_{0j} \\
\beta_{1j} &= \gamma_{10}
\end{align*}$$

where $U_{0j} \sim N(0, \tau_0^2)$ and independent (and independent w/r/t $R_{ij}$).
Random Intercept: 1 x

The appropriate diagram to represent this model?

\[ \text{SES}_{ij} \rightarrow \text{math}_{ij} \]

or more generally

\[ \text{x}_{ij} \rightarrow \text{y}_{ij} \]
Random Intercept: 1 x

The Linear Mixed Model:

\[ Y_{ij} = \gamma_{00} + \gamma_{10}x_{ij} + U_{0j} + R_{ij} \]

where

- \( \gamma_{00} \) is the intercept for the average group.
- \( \gamma_{10} \) is regression coefficient for \( x_{ij} \) (fixed).
- \( U_{0j} \sim \mathcal{N}(0, \tau_0^2) \) and independent.
- \( R_{ij} \sim \mathcal{N}(0, \sigma^2) \) and independent.
- \( U_{0j} \) and \( R_{ij} \) are independent.

Notes:

- \( \text{var}(Y_{ij}|x_{ij}) = \text{var}(U_{0j}) + \text{var}(R_{ij}) = \tau_0^2 + \sigma^2 \)
- \( \text{cov}(Y_{ij}, Y_{i'j}|x_{ij}, x_{i'j}) = \tau_0^2 \)
- Residual intra-class correlation,

\[ \rho_I(Y|X = x) = \frac{\tau_0^2}{\tau_0^2 + \sigma^2} \]

- If \( \tau_0^2 = 0 \), then simply use ordinary linear regression (OLS).
Marginal vs Hierarchical Model

The **Marginal Model**:

\[ Y_{ij} \sim N(\gamma_0 + \gamma_{10}x_{ij}, (\tau_o^2 + \sigma^2)) \]

- The hierarchical model implies the marginal model.
- The marginal model does **not** imply the hierarchical model.
- There are no random effects in the marginal model.
- The HLM is **more restrictive** than the marginal.
The linear mixed model,

\[ Y_{ij} = \gamma_{00} + \gamma_{10}x_{ij} + U_{0j} + R_{ij}, \]

In matrix notation, for marco unit (group) \( j \),

\[
\begin{pmatrix}
Y_{1j} \\
Y_{2j} \\
\vdots \\
Y_{nj,j}
\end{pmatrix}
= 
\begin{pmatrix}
1 & x_{1j} \\
1 & x_{2j} \\
\vdots & \vdots \\
1 & x_{nj,j}
\end{pmatrix}
\begin{pmatrix}
\gamma_{00} \\
\gamma_{10}
\end{pmatrix}
+ 
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}
( U_{0j} )
+ 
\begin{pmatrix}
R_{1j} \\
R_{2j} \\
\vdots \\
R_{nj,j}
\end{pmatrix}
\]

\[
Y_j = X_j \Gamma + Z_j U_j + R_j
\]
And the marginal model is

$$Y_j \sim \mathcal{N}(X_j \Gamma, (Z_j T Z_j' + \sigma^2 I)).$$

Note that $Z_j T Z_j'$ is just $(n_j \times n_j)$

$$1\tau^2 1' = \begin{pmatrix} \tau^2 & \tau^2 & \ldots & \tau^2 \\ \tau^2 & \tau^2 & \ldots & \tau^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tau^2 & \tau^2 & \ldots & \tau^2 \end{pmatrix}$$
Example: HSB

\[ x_{ij} = \text{SES}_{ij} \text{ (student/micro level)}. \]

Some descriptive statistics:

```
The MEANS Procedure

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics achievement</td>
<td>7185</td>
<td>2.75</td>
<td>6.88</td>
</tr>
<tr>
<td>Standardized student ses</td>
<td>7185</td>
<td>0.00</td>
<td>0.78</td>
</tr>
</tbody>
</table>
```
Edited output from SAS/MIXED

The Mixed Procedure

Model Information

<table>
<thead>
<tr>
<th>Data Set</th>
<th>SASDATA.HSBALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent Variable</td>
<td>mathach</td>
</tr>
<tr>
<td>Covariance Structure</td>
<td>Unstructured</td>
</tr>
<tr>
<td>Subject Effect</td>
<td>id</td>
</tr>
<tr>
<td>Estimation Method</td>
<td>ML</td>
</tr>
<tr>
<td>Residual Variance Method</td>
<td>Profile</td>
</tr>
<tr>
<td>Fixed Effects SE Method</td>
<td>Model-Based</td>
</tr>
<tr>
<td>Degrees of Freedom Method</td>
<td>Containment</td>
</tr>
</tbody>
</table>
Example: HSB

Dimensions

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Covariance Parameters</td>
<td>2</td>
</tr>
<tr>
<td>Columns in X</td>
<td>2</td>
</tr>
<tr>
<td>– Design for fixed</td>
<td></td>
</tr>
<tr>
<td>Columns in Z Per Subject</td>
<td>14</td>
</tr>
<tr>
<td>– Design for random</td>
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</tr>
<tr>
<td>Subjects</td>
<td>160</td>
</tr>
<tr>
<td>– N</td>
<td></td>
</tr>
<tr>
<td>Max Obs Per Subject</td>
<td>67</td>
</tr>
<tr>
<td>– largest $n_j$</td>
<td></td>
</tr>
<tr>
<td>Observations Used</td>
<td>7185</td>
</tr>
<tr>
<td>– $n_\pm$</td>
<td></td>
</tr>
<tr>
<td>Observations Not Used</td>
<td>0</td>
</tr>
<tr>
<td>Total Observations</td>
<td>7185</td>
</tr>
</tbody>
</table>

Convergence criteria met.
## Proc MIXED Output

### Covariance Parameter Estimates

<table>
<thead>
<tr>
<th>Cov Parm</th>
<th>Subject</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Z</th>
<th>Pr &gt; Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>UN(1,1)</td>
<td>id</td>
<td>4.7268</td>
<td>0.6483</td>
<td>7.29</td>
<td>&lt; .0001</td>
</tr>
<tr>
<td>Residual</td>
<td></td>
<td>37.0301</td>
<td>0.6253</td>
<td>59.22</td>
<td>&lt; .0001</td>
</tr>
</tbody>
</table>

### Solution for Fixed Effects

| Effect   | Estimate | Standard Error | DF | t Value | Pr > |t| |
|----------|----------|----------------|----|---------|------|---|
| Intercept| 12.6576  | 0.1873         | 159| 67.58   | < .0001|
| ses      | 2.3916   | 0.1057         | 7024| 22.63   | < .0001|
## Summary

<table>
<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add SES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>SE</td>
</tr>
<tr>
<td><strong>Fixed effects</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_{00}$</td>
<td>12.64</td>
<td>.24</td>
</tr>
<tr>
<td>$\gamma_{10}$</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td><strong>Random effects</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau_0^2$</td>
<td>8.55</td>
<td>1.07</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>39.15</td>
<td>.66</td>
</tr>
</tbody>
</table>

Residual intra-class correlation,

$$\hat{\rho}_I(\text{math}|\text{SES}) = \frac{4.73}{4.73 + 37.03} = \frac{4.73}{41.76} = .11$$
Comparison

- Drop in between groups variance estimate $\hat{\tau}^2_0$:

$$\frac{4.73}{8.55} = .55 \quad \text{or} \quad (1 - .56)100 = 45\% \text{ decrease}$$

- Drop in within groups variance estimate $\hat{\sigma}^2$:

$$\frac{37.03}{39.15} = .95 \quad \text{or} \quad (1 - .95)100 = 5\% \text{ decrease}$$

Interpretation somewhat problematic because SES helps to explain both the between and within groups variance of $Y_{ij}$:

$$\text{SES}_{ij} = \overline{\text{SES}}_j + (\text{SES}_{ij} - \overline{\text{SES}}_j)$$
Problematic Interpretation

...Between versus Within Group Regressions.
Different processes at work at different levels.

An example:
Problematic Interpretation (continued)

Within individual: Child’s reading ability improves with instruction.

Between individuals: Children get tutoring/private instruction for different reasons.
Returning to the HSB Example

Groups differ with respect to mean SES (i.e., $\bar{\text{SES}}_j$).

HSB: Group Mean SES by School ID
Returning to the HSB Example

Individual students within schools differ with respect to SES.

HSB: Student SES by School ID
Returning to the HSB Example

Or for just one school, we have within school variability,

Line is simple linear regression line.
Solution to Problem

Regarding different mean levels of SES between and within schools:

- “Group mean centered” variable, e.g.,

\[ x_{ij} = (\text{SES}_{ij} - \overline{\text{SES}}_j) \]

...to model within group variability of \( Y_{ij} \) w/rt SES.

- Group mean as a level 2 (school) variable

\[ z_j = \overline{\text{SES}}_j \]
Recall that overall mean SES = 0.

First we’ll just have a level 1 (student) variable for SES:

\[ x_{ij} = (SES_{ij} - \overline{SES}_j) \]

A diagram for this model,

\[ (SES_{ij} - \overline{SES}_j) \rightarrow \text{math}_{ij} \]
Add Centered SES at Level 1?
Some More Schools
Overlay Regressions for all Schools

Math x Homework
Separate Regression Line for Each School
Hierarchical Model with Centered SES

Level 1:

\[
Y_{ij} = \beta_0j + \beta_1j x_{ij} + R_{ij}
\]

\[
\text{math}_{ij} = \beta_0j + \beta_1j (\text{SES}_{ij} - \overline{\text{SES}_j}) + R_{ij}
\]

where \( R_{ij} \sim N(0, \sigma^2) \) and independent.

Level 2:

\[
\beta_0j = \gamma_{00} + U_{0j}
\]

\[
\beta_1j = \gamma_{10}
\]

where \( U_{0j} \sim N(0, \tau_0^2) \) and independent.
Linear Mixed & Marginal Models

The corresponding linear mixed model:

\[ \text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_{j}) + U_{0j} + R_{ij} \]

and Marginal Model:

\[ \text{math}_{ij} \sim \mathcal{N}(\gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_{j}), (\tau_{0}^{2} + \sigma^{2})) \].
**Edited SAS/MIXED Output**

Convergence criteria met.

### Covariance Parameter Estimates

<table>
<thead>
<tr>
<th>Cov Parm</th>
<th>Subject</th>
<th>Estimate</th>
<th>Error</th>
<th>Value</th>
<th>Pr &gt; Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>UN(1,1) id</td>
<td></td>
<td>8.6071</td>
<td>1.0682</td>
<td>8.06</td>
<td>&lt; .0001</td>
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<tr>
<td>Residual</td>
<td></td>
<td>37.0056</td>
<td>0.6245</td>
<td>59.26</td>
<td>&lt; .0001</td>
</tr>
</tbody>
</table>

### Solution for Fixed Effects

| Effect   | Estimate | Error | DF  | Value | Pr > |t| |
|----------|----------|-------|-----|-------|------|---|
| Intercept| 12.6494  | 0.2437| 159 | 51.92 | < .0001|
| cses    | 2.1912   | 0.1086| 7024| 20.17 | < .0001|
Estimated Overall Regression Line

\[ \hat{\text{math}}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) \]

Overall Regression

Graph showing the fitted math scores against group mean centered SES.
Sub-Sample of Groups’ Lines

\[ \hat{\text{math}}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + \hat{U}_j \]

and Some of the 160 Schools

![Graph showing fitted math scores against group mean centered SES]
## Model Summary & Comparison

<table>
<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add SES</th>
<th>Add cSES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fixed effects</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_{00}$</td>
<td>12.64</td>
<td>12.66</td>
<td>12.65</td>
</tr>
<tr>
<td>$\gamma_{10}$</td>
<td>—</td>
<td>2.39</td>
<td>2.19</td>
</tr>
<tr>
<td><strong>Random effects</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau_0^2$</td>
<td>8.55</td>
<td>4.73</td>
<td>8.61</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>39.15</td>
<td>37.03</td>
<td>37.01</td>
</tr>
</tbody>
</table>
Conclusion

Residual intra-class correlation,

\[ \hat{\rho}_I(\text{math}|\text{SES}) = \frac{8.61}{8.61 + 37.01} = \frac{8.61}{45.62} = .19 \]

It appears that SES account for some variability in math achievement, \( Y_{ij} \), within schools

\[ \left(1 - \frac{37.01}{39.15}\right) \times 100\% = (1 - .945)100\% = 5.5\% \]

Does \( z_j = \overline{SES}_j \) help model between group variability?
Between School Variability & SES

Using $\overline{SES}_j$ as predictor for $\beta_{oj}$.

The $\hat{U}_{0j}$'s plotted below are from

$$Y_{ij} = (\gamma_{00} + U_{0j}) + \gamma_{10}(SES_{ij} - \overline{SES}_j) + R_{ij}$$

$U_{0j} = .033074 + 5.344104 \times \text{(mean SES)}_j$

$r = .789$ and $R^2 = .623$
HLM with $x$ and $z$

Level 1:

$$\text{math}_{ij} = \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + R_{ij}$$

where $R_{ij} \sim N(0, \sigma^2)$.

Level 2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01}(\overline{\text{SES}}_j) + U_{0j}$$
$$\beta_{1j} = \gamma_{10}$$

where $U_{0j} \sim N(0, \tau_0^2)$ and independent.

Linear Mixed Model

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \gamma_{01}(\overline{\text{SES}}_j) + U_{0j} + R_{ij}$$
Guessing What Will Happen

\[ \text{math}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + U_{0j} + R_{ij} \]

Put in regression of \( \hat{U}_{0j} \) on \( \overline{\text{SES}}_j \) into above:

\[ Y_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + (.0331 + 5.344(\overline{\text{SES}}_j) + U_{0j}^*) + R_{ij} \]

\[ = 12.6825 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + 5.344(\overline{\text{SES}}_j) + U_{0j}^* + R_{ij} \]

How about a new \( \tau_0^* \)?

The \( R^2 \) between \( \hat{U}_{0j} \) & \( \overline{\text{SES}}_j \) equaled .623 and previously estimated \( \hat{\tau}_0 = 8.6071 \).

\[ \frac{\hat{\tau}_0 - \hat{\tau}_0^*}{\hat{\tau}_0} \approx R^2 \]

\[ \frac{8.6071 - \hat{\tau}_0^*}{8.6071} \approx .6233 \rightarrow \hat{\tau}_0^* \approx 3.24 \]
Theory and Proposition

In terms of a diagram, our proposition (theory),

$$\overline{SES}_j$$

$$(SES_{ij} - \overline{SES}_j) \rightarrow \text{math}_{ij}$$

**Micro-micro:** Higher a student’s SES relative to the school mean, the higher his/her math score.

**Macro-micro:** Higher average SES of school, higher a student’s math score.
Edited SAS/MIXED Output

Dimensions

Covariance Parameters   2
Columns in X           3  *** this changed
Columns in Z Per Subject 1
Subjects                160
Observations Used       7185
Observations Not Used   0
Total Observations      7185

Convergence criteria met.
## Edited SAS/MIXED Output

### Covariance Parameter Estimates

<table>
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<tr>
<th>Cov Parm</th>
<th>Sub</th>
<th>Estimate</th>
<th>Error</th>
<th>Value</th>
<th>Pr &gt; Z</th>
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### Solution for Fixed Effects

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<th>Error</th>
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<th>Value</th>
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</table>
Summary Models Fit to HSB

**Null**, empty, baseline,

\[ \text{math}_{ij} = \gamma_{00} + U_{0j} + R_{ij} \]

1 uncentered variable,

\[ \text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES})_{ij} + U_{0j} + R_{ij} \]

Group centered,

\[ \text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}_j}) + U_{0j} + R_{ij} \]

Group centered and mean,

\[ \text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}_j}) + \gamma_{01}(\overline{\text{SES}_j}) + U_{0j} + R_{ij} \]
### Summary of Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add (SES) (_{ij})</th>
<th>Add cSES(_{ij}) &amp; (SES)(_j)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>SE</td>
<td>Value</td>
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<tr>
<td>(\gamma_{00})</td>
<td>12.64</td>
<td>.24</td>
<td>12.66</td>
</tr>
<tr>
<td>(\gamma_{10})</td>
<td>—</td>
<td>—</td>
<td>2.39</td>
</tr>
<tr>
<td>(\gamma_{01})</td>
<td>—</td>
<td>—</td>
<td>—</td>
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<td><strong>Random effects</strong></td>
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<td>(\tau^2_0) (between)</td>
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<tr>
<td>(\sigma^2) (within)</td>
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<td>.66</td>
<td>37.03</td>
</tr>
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</table>

**Note:**

\[cSES_{ij} = (SES)_{ij} - (\overline{SES})_j\]

\[\overline{SES}_j = (1/n_j) \sum_{i=1}^{n_j} (SES)_{ij}\]
Two more models

- Add all the student level variables that we have to see if we can account for more of the within groups residual variance.
  
  i.e., Add more $x_{ij}$’s to the model.
  
  - $\text{SES}_{ij} - \overline{\text{SES}}_j$
  - female ($= 1$ if female, $= 0$ if male)
  - minority ($= 1$ if minority, $= 0$ if not)

- Add all the school level variables to see if we can account for more between groups residual variance.
  
  i.e., Add more $z_j$’s to the model.
  
  - himinty = 40% minority, 0 $<=$ 40% minority
  - pracad = Proportion of students in academic track
  - disclim = Disciplinary climate
  - sector $= 1$ = Catholic, 0 = public
  - size = school enrollment
HLM with Lots of Micro Variables

\[ \text{Level 1:} \quad (\text{math})_{ij} = \beta_0 + \beta_1 (\text{SES}_{ij} - \overline{\text{SES}}_j) + \beta_2 (\text{female})_{ij} + \beta_3 (\text{minority})_{ij} + R_{ij} \]

where \( R_{ij} \sim \mathcal{N}(0, \sigma^2) \) and independent.

\[ \text{Level 2:} \quad \beta_0 = \gamma_{00} + \gamma_{01} \overline{\text{SES}}_j + U_{0j} \]
\[ \beta_1 = \gamma_{10} \]
\[ \beta_2 = \gamma_{20} \]
\[ \beta_3 = \gamma_{30} \]

where \( U_{0j} \sim \mathcal{N}(0, \tau_0^2) \) and independent. \( R_{ij} \) and \( U_j \) are independent.

**Homework:** What's the linear mixed model? What's the marginal model?
Edited SAS/MIXED output

The Mixed Procedure

Model Information

Data Set                  WORK.HSBCENT
Dependent Variable        mathach
Covariance Structure      Unstructured
Subject Effect            id
Estimation Method         ML
Residual Variance Method  Profile
Fixed Effects SE Method   Model-Based
Degrees of Freedom Method Containment
Edited SAS/MIXED output

Dimensions

<table>
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<tr>
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<tr>
<td>Columns in Z Per Subject</td>
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<td>Subjects</td>
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<tr>
<td>Max Obs Per Subject</td>
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<tr>
<td>Observations Used</td>
<td>7185</td>
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<td>Observations Not Used</td>
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<tr>
<td>Total Observations</td>
<td>7185</td>
</tr>
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</table>
Edited SAS/MIXED output

Convergence criteria met.

Covariance Parameter Estimates

<table>
<thead>
<tr>
<th>Cov Parm</th>
<th>Subject</th>
<th>Estimate</th>
<th>Error</th>
<th>Value</th>
<th>Pr Z</th>
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</table>
### Solution for Fixed Effects

<table>
<thead>
<tr>
<th>Effect</th>
<th>Student</th>
<th>gender</th>
<th>Standard Error</th>
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<tbody>
<tr>
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<tr>
<td>female 0</td>
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<tr>
<td>female 1</td>
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<td>meanses</td>
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</table>
## Summary & Comparison

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<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add cSES$<em>{ij}$ &amp; (SES)$</em>{j}$</th>
<th>Add all micro level</th>
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<td><strong>Fixed effects</strong></td>
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<tr>
<td>intercept</td>
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<td>12.65</td>
<td>12.66</td>
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<td>—</td>
</tr>
<tr>
<td>not minority</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>mean SES</td>
<td>—</td>
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<td>5.87</td>
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<td></td>
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<td>10.10</td>
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<td><strong>Random effects</strong></td>
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<td></td>
</tr>
<tr>
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<tr>
<td>$\sigma^2$ (within)</td>
<td>39.15</td>
<td>37.01</td>
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</table>

Note:

\[
\begin{align*}
\text{cSES}_{ij} &= (\text{SES})_{ij} - (\text{SES})_{j} \\
\text{SES}_{j} &= (1/n_j) \sum_{i=1}^{n_j} (\text{SES})_{ij}
\end{align*}
\]
Lots of Macro-level Variables

Add all of the school (macro-level) variables:

- himinty = 40% minority, 0 = < 40% minority
- pracad = Proportion of students in academic track
- disclim = Disciplinary climate
- sector = 1 = Catholic, 0 = public
- size = school enrollment
HLM for Lots of Macro-level

Level 1:

$$(\text{math})_{ij} = \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \beta_{2j}(\text{female})_{ij} + \beta_{3j}(\text{minority})_{ij} + R_{ij}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

Level 2:

$\beta_{0j} = \gamma_{00} + \gamma_{01}(\overline{\text{SES}})_j + \gamma_{02}(\text{himinty})_j + \gamma_{03}(\text{pracad})_j + \gamma_{04}(\text{disclim})_j + \gamma_{05}(\text{sector})_j + \gamma_{06}(\text{size})_j + U_{0j}$

$\beta_{1j} = \gamma_{10}$

$\beta_{2j} = \gamma_{20}$

$\beta_{3j} = \gamma_{30}$

where $U_{0j} \sim \mathcal{N}(0, \tau^2_0)$ and independent, and $R_{ij}$ and $U_j$ are independent.
Linear Mixed w/ Lots of Macro-level

\[
(math)_{ij} = \gamma_0 + \gamma_{10}(SES_{ij} - \overline{SES}_j) + \gamma_{20}(female)_{ij} \\
+ \gamma_{30}(minority)_{ij} \\
+ \gamma_{01}(\overline{SES})_j + \gamma_{02}(himinty)_j + \gamma_{03}(pracad)_j \\
+ \gamma_{04}(disclim)_j + \gamma_{05}(sector)_j + \gamma_{06}(size)_j \\
+ U_{0j} + R_{ij}
\]

What’s the marginal model?
**Edited SAS/MIXED output**

The Mixed Procedure

**Dimensions**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
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<td>Columns in Z Per Subject</td>
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<tr>
<td>Subjects</td>
<td>160</td>
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<tr>
<td>Max Obs Per Subject</td>
<td>67</td>
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<tr>
<td>Observations Used</td>
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<td>Observations Not Used</td>
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<tr>
<td>Total Observations</td>
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</table>
**Convergence criteria met.**

**Covariance Parameter Estimates**

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<tr>
<th>Cov Parm</th>
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<th>Estimate</th>
<th>Standard Error</th>
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## Edited SAS/MIXED output: Fixed Effects

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## Edited SAS/MIXED output

### Type 3 Tests of Fixed Effects

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<td>size</td>
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<td>7022</td>
<td>12.07</td>
<td>0.0005</td>
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## Summary & Comparison: Fixed Effects

<table>
<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add cSES&lt;sub&gt;ij&lt;/sub&gt; &amp; (SES)&lt;sub&gt;j&lt;/sub&gt;</th>
<th>Add all micro level</th>
<th>All micro and macro</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>SE</td>
<td>Value</td>
<td>SE</td>
</tr>
<tr>
<td>intercept</td>
<td>12.64</td>
<td>.24</td>
<td>12.66</td>
<td>.15</td>
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<tr>
<td>centered SES</td>
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<td></td>
<td>2.19</td>
<td>.11</td>
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<tr>
<td>male</td>
<td></td>
<td></td>
<td>1.22</td>
<td>.16</td>
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<tr>
<td>not minority</td>
<td></td>
<td></td>
<td>2.73</td>
<td>.20</td>
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<tr>
<td>mean SES</td>
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<td></td>
<td>5.87</td>
<td>.36</td>
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<td>himinty</td>
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<tr>
<td>sector</td>
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<tr>
<td>size</td>
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</tr>
</tbody>
</table>

Note: cSES<sub>ij</sub> = (SES)<sub>ij</sub> − (SES)<sub>j</sub>

\[ \text{SES}_j = \sum_{i=1}^{n_j} (SES)_{ij} \]
## Summary & Comparison: Random

<table>
<thead>
<tr>
<th></th>
<th>Null Model</th>
<th>Add cSES&lt;sub&gt;ij&lt;/sub&gt; &amp; (SES)&lt;sub&gt;j&lt;/sub&gt;</th>
<th>Add all micro level</th>
<th>All micro and macro</th>
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<tr>
<td><strong>Random effects</strong></td>
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<tr>
<td>( \tau_0^2 ) (between)</td>
<td>8.55 1.07</td>
<td>2.64 .40</td>
<td>2.39 .37</td>
<td>1.29 .24</td>
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<tr>
<td>( \sigma^2 ) (within)</td>
<td>39.15 .66</td>
<td>37.02 .62</td>
<td>35.89 .61</td>
<td>35.88 .61</td>
</tr>
</tbody>
</table>

Note:
\[
cSES_{ij} = (SES)_{ij} - (\overline{SES})_j \\
\overline{SES}_j = \sum_{i=1}^{n_j} (SES)_{ij}
\]
Summary: Random Intercept Models

Concepts covered:

- Within group dependency accounted for by random effect (i.e., $R_{ij}$).
- Between group differences accounted for by random intercept (i.e., $U_{0j}$).
- Central role of variance between and within groups.
- Effect of adding Level 1 and Level 2 predictors.
- Mean centering of level 1 predictors and adding the mean back in at Level 2.
- Intra-class correlation or $ICC$.
- HLMs with only random intercepts imply homoscedasticity.